

Topological aspects of generalized Harper operators

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Abstract. A generalized version of the TKNN-equations computing Hall conductances for generalized Dirac-like Harper operators is derived. Geometrically these equations relate Chern numbers of suitable (dual) bundles naturally associated to spectral projections of the operators.

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GENERALIZED HARPER OPERATORS

The *integer quantum Hall effect* (IQHE) reveals a variety of surprising and attractive physical features, and has been the subject of several investigations (see [17, 13] and references therein). In fact, a complete spectral analysis of the Schrödinger operator for a single particle moving in a plane in a periodic potential and subject to an uniform orthogonal magnetic field of strength B (*magnetic Bloch electron*) is extremely difficult. Thus the need for simpler effective models which hopefully capture (some of) the main physical features in suitable physical regimes.

In the limit of a strong magnetic field, $B \gg 1$, the IQHE is well described by an effective Harper operator (cf. [23, 2, 15, 12]). For this model the quantization (in units of e^2/h) of the Hall conductance has a geometric meaning being related to Chern numbers of suitable naturally bundles associated to spectral projections of the operator. A family of Diophantine equations, the TKNN-equations of [22], provides a recipe for computing such integers. The aim of the present paper is to derive a generalized version of the TKNN-equations yielding the Hall conductances for more general Dirac-like Harper operators. Interest in such generalizations comes also from these Dirac-like operators appearing naturally in important physical models, notably models for the graphene.

With $\theta = 1/B$, the effective Harper operator is

$$(H_{1,0}^\theta \psi)(x) = \psi(x - \theta) + \psi(x + \theta) + 2 \cos(2\pi x) \psi(x), \quad (1)$$

acting on the Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R})$. That this operator is the simplest representative of a large family of generalized Harper operators, sharing similar mathematical properties, is our starting point. On the Hilbert space \mathcal{H}_1 consider the unitary operators

$$(T_1 \psi)(x) = e^{i2\pi x} \psi(x), \quad (T_2^\theta \psi)(x) = \psi(x - \theta), \quad (2)$$

with $\theta \in \mathbb{R}$. They are readily seen to obey the relation

$$T_1 T_2^\theta = e^{i2\pi\theta} T_2^\theta T_1, \quad (3)$$

yielding for the Harper operator the expression $H_{1,0}^\theta = T_1 + (T_1)^\dagger + T_2^\theta + (T_2^\theta)^\dagger$.

For any positive integer $q = 1, 2, \dots$, on the vector space \mathbb{C}^q consider two unitary $q \times q$ matrices \mathbb{U}_q and \mathbb{V}_q defined as follows. Let $\{e_0, \dots, e_{q-1}\}$ be the canonical basis of \mathbb{C}^q , then \mathbb{U}_q is a diagonal matrix and \mathbb{V}_q is a shift matrix acting as

$$\mathbb{U}_q : e_j \mapsto e^{i2\pi \frac{j}{q}} e_j, \quad \text{and} \quad \mathbb{V}_q : e_j \mapsto e_{[j+1]_q},$$

where $[\cdot]_q$ stays for modulo q . They obey

$$\mathbb{U}_q \mathbb{V}_q = e^{i2\pi \frac{1}{q}} \mathbb{V}_q \mathbb{U}_q \quad \text{and} \quad (\mathbb{U}_q)^q = \mathbb{I}_q = (\mathbb{V}_q)^q.$$

Then, on the Hilbert space $\mathcal{H}_q = L^2(\mathbb{R}) \otimes \mathbb{C}^q$ one defines a pair of unitary operators:

$$U_q = T_1 \otimes \mathbb{U}_q, \quad V_{q,r}^\theta = T_2^\varepsilon \otimes (\mathbb{V}_q)^r, \quad (4)$$

with $\varepsilon(\theta, q, r) = \theta - \frac{r}{q}$ and T_1 and T_2^ε given by (2). The integer $r \in \{0, \pm 1, \dots, \pm(q-1)\}$ is chosen coprime with respect to q . As for the case before (when $q = 1, r = 0$), the operators (4) also obey the relation (3):

$$U_q V_{q,r}^\theta = e^{i2\pi\theta} V_{q,r}^\theta U_q.$$

Following the definition (1) we can introduce the *generalized (q, r) -Harper operator*

$$H_{q,r}^\theta = U_q + (U_q)^\dagger + V_{q,r}^\theta + (V_{q,r}^\theta)^\dagger. \quad (5)$$

More generally one considers the collection $\mathcal{A}_{q,r}^\theta$ of bounded operators on the Hilbert space \mathcal{H}_q generated by the unitaries U_q and $V_{q,r}^\theta$. Technically $\mathcal{A}_{q,r}^\theta$ is a C^* -algebra, i.e. an involutive algebra closed with respect to the operator norm, and it is named the *(ir)rational rotation algebra* or the *noncommutative torus algebra* [19, 8].

By writing the operator in (1) as $H_{1,0}^\theta = D_\theta + C$ with

$$(D_\theta \psi)(x) = \psi(x - \theta) + \psi(x + \theta), \quad (C\psi)(x) = 2 \cos(2\pi x) \psi(x), \quad (6)$$

in particular, the generalized $(2, 1)$ -Harper operator $H_{2,1}^\theta$ is just

$$H_{2,1}^\theta = \begin{pmatrix} C & D_{\theta-\frac{1}{2}} \\ D_{\theta-\frac{1}{2}} & -C \end{pmatrix} \quad (7)$$

acting on $\mathcal{H}_2 = L^2(\mathbb{R}) \otimes \mathbb{C}^2$. This operator provides an interesting effective model for the IQHE on graphene [3, 14, 21]. Moreover, Dirac-like operators like $H_{2,1}^\theta$ can be used to describe effective models for electrons interacting with the periodic structure of a crystal through a periodic (internal) magnetic field and subjected to the action of an external strong magnetic field [9, 12].

BLOCH-FLOQUET TRANSFORM

For rational deformation parameter, $\theta = M/N$ (M and N taken to be coprime here and after), any family of operators $\mathcal{A}_{q,r}^\theta$ can be decomposed in a continuous way according to a generalized version of the Bloch theorem. More explicitly we have the following.

Proposition A. *Let $\theta = M/N$. For any (admissible) pair (q, r) the bounded operator algebra $\mathcal{A}_{q,r}^\theta$ on \mathcal{H}_q admits a bundle representation $\Pi_{q,r}$ over the ordinary two-torus \mathbb{T}^2 . That is to say, there is a Hermitian vector bundle $E_{N,q} \rightarrow \mathbb{T}^2$ together with a unitary transform $\mathcal{F}_{q,r} : \mathcal{H}_q \rightarrow L^2(E_{N,q})$ such that*

$$\Pi_{q,r}(\mathcal{A}_{q,r}^\theta) = \mathcal{F}_{q,r} \mathcal{A}_{q,r}^\theta \mathcal{F}_{q,r}^{-1} \subset \Gamma(\text{End}(E_{N,q})). \quad (8)$$

The vector bundle $E_{N,q}$ has rank N and (first) Chern number $C_1(E_{N,q}) = q$.

Here $L^2(E_{N,q})$ denotes the Hilbert space of square integrable sections of the vector bundle $E_{N,q}$ and $\Gamma(\text{End}(E_{N,q}))$ denotes the collection of continuous sections of the endomorphism bundle $\text{End}(E_{N,q}) \rightarrow \mathbb{T}^2$, i.e. the vector bundle with fibers $\text{End}(\mathbb{C}^q)$ associated with the vector bundle $E_{N,q}$. The unitary map $\mathcal{F}_{q,r}$ implementing the bundle representation of $\mathcal{A}_{q,r}^\theta$ is called (*generalized*) *Bloch-Floquet transform* [9, 11]. For the details of the proof of Prop. A, that we briefly sketch, we refer to [10] (see also [20]). Denote with $\alpha \in \mathbb{Z}$, $|\alpha| < q$ the unique solution of $\beta q - \alpha r = 1$ (due to q and r being coprime) and be $M_0 = qM - rN$. Then, a simple check shows that the unitary operators

$$A_{q,r}^\theta = (T_1)^{\frac{1}{q\epsilon}} \otimes (\mathbb{U}_q)^\alpha, \quad B_{q,r}^\theta = T_2^{\frac{M_0}{q}} \otimes (\mathbb{V}_q)^{rN}$$

commute, $[A_{q,r}^\theta, B_{q,r}^\theta] = 0$, while commuting with any element in $\mathcal{A}_{q,r}^\theta$. They generate a (indeed maximally) commutative sub-algebra of the commutant of $\mathcal{A}_{q,r}^\theta$, and in particular, of symmetries for the operator (5). Were N a multiple of q this commutative sub-algebra would reduce to a direct sum of q copies of a commutative algebra on $L^2(\mathbb{R})$. Thus to avoid this degeneracy, we take q and N to be coprime as well. This entails there exist two integers d_r and n_r such that $qd_r + Nn_r = 1$, a fact we shall exploit momentarily. Moreover, the commutative algebra generated by $A_{q,r}^\theta$ and $B_{q,r}^\theta$ is isomorphic to the algebra of continuous functions over the ordinary 2-torus \mathbb{T}^2 .

A generalized simultaneous eigenvectors of $A_{q,r}^\theta$ and $B_{q,r}^\theta$ is a $\Xi_k \in S'(\mathbb{R}) \otimes \mathbb{C}^q$ ($S'(\mathbb{R})$ is the space of tempered distributions) such that

$$A_{q,r}^\theta \Xi_k = e^{i2\pi k_1} \Xi_k, \quad B_{q,r}^\theta \Xi_k = e^{i2\pi k_2} \Xi_k.$$

For any $k = (k_1, k_2) \in [0, 1]^2 \simeq \mathbb{T}^2$, the generalized eigenvectors make up a N -dimensional space, a basis of which being given by a fundamental family of distribution $\Upsilon^{(j)}(k) = (\zeta_0^{(j)}(k), \dots, \zeta_{q-1}^{(j)}(k)) \in S'(\mathbb{R}) \otimes \mathbb{C}^q$, for indices $j = 0, \dots, N-1$, with elements $\zeta_\ell^{(j)}(k)$, $\ell = 0, \dots, q-1$, defined by

$$\zeta_\ell^{(j)}(k) = \sqrt{\frac{|M_0|}{N}} \sum_{m \in \mathbb{Z}} e^{-i2\pi k_1(\tau_\ell + mq)} \delta \left[\cdot - \frac{M_0}{N}(k_2 + j) - mM_0 - \tau_\ell \frac{M_0}{q} \right]. \quad (9)$$

Here the permutation $\tau : \ell \mapsto \tau_\ell$ of the set $\{0, \dots, q-1\}$ is defined by $\ell = [\tau_\ell r N]_q$ and, as usual, the Dirac delta function $\delta(\cdot - x_0)$ acts on functions $f : \mathbb{R} \rightarrow \mathbb{C}$ as the evaluation at the point x_0 , i.e. $\langle \delta(\cdot - x_0); f \rangle = f(x_0)$.

We let $\mathcal{H}_{q,r}(k) \subset \mathcal{S}'(\mathbb{R}) \otimes \mathbb{C}^q$ denote the N -dimensional vector space spanned by the distributions $\Upsilon^{(0)}(k), \dots, \Upsilon^{(N-1)}(k)$. The total space of the vector bundle $E_{N,q}$ is just the disjoint union of the spaces $\mathcal{H}_{q,r}(k)$ glued together with transition functions coming from pseudo-periodic conditions satisfied by the $\Upsilon^{(j)}$'s. Indeed, from (9) one deduces that $\Upsilon^{(j)}(k_1 + 1, k_2) = \Upsilon^{(j)}(k_1, k_2)$ while $\Upsilon^{(j)}(k_1, k_2 + 1) = \Upsilon^{(j+1)}(k_1, k_2)$ for $j = 0, \dots, N-2$ and $\Upsilon^{(N-1)}(k_1, k_2 + 1) = e^{i2\pi q} \Upsilon^{(0)}(k_1, k_2)$. Also, there is an identification

$$\mathcal{F}_{q,r} : \mathcal{H}_q \rightarrow L^2(E_{N,q}) \simeq \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}_{q,r}(k) \, dz(k)$$

which is very reminiscent of the usual direct integral decomposition of the Bloch theory. We stress that Prop. A does not only states that any $H \in \mathcal{A}_{q,r}^\theta$ can be decomposed as a direct integral operator $H = \int_{\mathbb{T}^2}^{\oplus} h(k) \, dz(k)$ with $h(k)$ an $N \times N$ matrix acting on $\mathcal{H}_{q,r}(k)$, but also that such a decomposition is continuous with respect to the topology of the vector bundle $E_{N,q}$, thus amounting to a bundle representation. For $H \in \mathcal{A}_{q,r}^\theta$, we denote $\tilde{H} = \Pi_{q,r}(H)$. For the generators, when acting on the basis $\{\Upsilon^{(j)}(k)\}$ one finds

$$\tilde{U}_q(k_1, k_2) = e^{i2\pi \frac{M_0}{N} k_2} (\mathbb{U}_N)^{qM}, \quad \tilde{V}_{q,r}^\theta(k_1, k_2) = e^{i2\pi n_r k_1} (\mathbb{V}_{N;k_1})^{dr}. \quad (10)$$

Here \mathbb{U}_N is the diagonal matrix $\mathbb{U}_N : e_j \mapsto e^{i2\pi \frac{j}{N}} e_j$; $\mathbb{V}_{N;k_1}$ is the twisted shift matrix sending e_j to e_{j+1} for $j = 0, \dots, N-2$ while e_{N-1} to $e^{i2\pi q k_1} e_0$. The matrices in (10) commute up to $e^{i2\pi \frac{M}{N} q dr} = e^{i2\pi \frac{M}{N}}$ being $qdr = 1 - n_r N$ as before, $\tilde{U}_q \tilde{V}_{q,r}^\theta = e^{i2\pi \frac{M}{N}} \tilde{V}_{q,r}^\theta \tilde{U}_q$, thus providing a representation of $\mathcal{A}_{q,r}^\theta$. Moreover, their pseudo-periodic conditions

$$\tilde{U}_q(k_1 + 1, k_2 + 1) = e^{i2\pi \frac{M}{N}} \tilde{U}_q(k_1, k_2), \quad \tilde{V}_{q,r}^\theta(k_1 + 1, k_2 + 1) = \tilde{V}_{q,r}^\theta(k_1, k_2),$$

match those of the basis $\{\Upsilon^{(j)}(k)\}$ thus making $\Pi_{q,r}$ a representation of $\mathcal{A}_{q,r}^\theta$ as bundle endomorphisms, as expressed in (8).

The bundle $E_{N,q}$ comes equipped with the *Berry connection*

$$\omega_{i,j}(k) = \langle \Upsilon^i(k) | d\Upsilon^j(k) \rangle, \quad i, j = 0, \dots, N-1, \quad (11)$$

Its curvature $K = d\omega$ is constant, $K(k) = \left(\frac{2\pi q}{iN} \mathbb{I}_N \right) dk_1 \wedge dk_2$ (up to an exact form) and when integrated it results in the first Chern number of the bundle being

$$C_1(E_{N,q}) = \frac{i}{2\pi} \int_{\mathbb{T}^2} \text{Tr}_N(K) = q.$$

GENERALIZED TKNN-EQUATIONS

For a rational $\theta = M/N$, the spectrum of $H_{q,r}^\theta$ in (5) has $N + 1$ energy bands if N is odd or N energy bands if N is even [16, 6, 10]. These include the *inf-gap* (from $-\infty$ to the minimum of the spectrum) and the *sup-gap* (from the maximum of the spectrum to $+\infty$).

To each gap g one associates a spectral projection P_g with the convention that $P_0 = 0$ for the inf-gap $g = 0$ and $P_{\max} = \mathbb{I}$ for the sup-gap $g = N_{\max}$ with $N_{\max} = N - 1$ or $N_{\max} = N$ according to whether N is odd or even. As usual, the projection P_g is defined via the Riesz formula for the operator $H_{q,r}^\theta$,

$$P_g = \frac{1}{i2\pi} \oint_{\Lambda} (\lambda \mathbb{I} - H_{q,r}^\theta)^{-1} d\lambda. \quad (12)$$

The closed rectifiable path $\Lambda \subset \mathbb{C}$ encloses the spectral subset $I_g = [\varepsilon_0, \varepsilon_g] \cap \sigma(H_{q,r}^\theta)$ (intersecting the real axis in ε_0 and ε_g) with the real numbers $\varepsilon_0, \varepsilon_g \in \mathbb{R} \setminus \sigma(H_{q,r}^\theta)$ being such that $-\infty < \varepsilon_0 < \min \sigma(H_{q,r}^\theta)$ and ε_g in the gap g .

The Hall conductance associated with the energy spectrum up to the gap g is related to the projection P_g via the Kubo formula (linear response theory). Its value is an integer number t_g ; it is by now well known that t_g is to be thought of as the Chern number of a bundle determined by the projection P_g [22, 4, 1].

Any such a spectral projection P_g yields a projection $\Pi_{q,r}(P_g) \in \Gamma(\text{End}(E_{N,q}))$, via the representation $\Pi_{q,r}$ in (8), and thus a vector subbundle $L_{q,r}(P_g) \subset E_{N,q}$. The related (first) Chern number $C_1(L_{q,r}(P_g))$ measures the degree of non triviality of the bundle $L_{q,r}(P_g)$. The geometric interpretation of the Hall conductance is none other than the equality $t_g = C_1(L_{q,r}(P_g))$. On the other hand, the Chern number $C_1(L_{q,r}(P_g))$ obeys a Diophantine equation which then provides a TKNN-type equation for the conductance t_g . We have the following.

Proposition B. *For any projection P in the algebra $\mathcal{A}_{q,r}^\theta$ there exists a “dual” vector bundle $L_{\text{ref}}(P) \rightarrow \mathbb{T}^2$ s.t. the following duality between Chern numbers holds:*

$$C_1(L_{q,r}(P)) = q \left[\frac{1}{N} \text{Rk}(L_{\text{ref}}(P)) + \left(\frac{M}{N} - \frac{r}{q} \right) C_1(L_{\text{ref}}(P)) \right]. \quad (13)$$

Before we sketch the proof of this result we turn to its interpretation in terms of conductances of the generalized Harper operators in (5). As mentioned, if P_g is its spectral projection up to the gap g , the associated Hall conductance t_g is the number $C_1(L_{q,r}(P_g))$. For the dual number we have $C_1(L_{\text{ref}}(P_g)) = -s_g$, with s_g identified with the Hall conductance of the energy spectrum up to the gap g but in the opposite limit of a weak magnetic field ($B \ll 1$) [22, 1, 9]. Writing $d_g = \text{Rk}(L_{\text{ref}}(P_g))$, relation (13) translates to the *generalized TKNN-equations*

$$N t_g + (qM - rN) s_g = q d_g, \quad g = 0, \dots, N_{\max}. \quad (14)$$

When $q = 1$ and $r = 0$, the above reduces to

$$N t_g + M s_g = d_g, \quad g = 0, \dots, N_{\max}, \quad (15)$$

which is the original TKNN-equation derived in [22] for the Harper operator (1). In its spirit then, the integer d_g in the right-hand side coincides with the labeling of the gap when N is odd, i.e. $d_g = g$ for N odd. When N is even $d_g = g$ if $0 \leq g \leq N/2 - 1$ and $d_g = g + 1$ if $N/2 \leq g \leq N_{\max} = N - 1$. We remark that the bound

$$2|s_g| < N \quad (16)$$

(already present in [22]) still holds, owing to the bound $2|C_1(L_{\text{ref}}(P_g))| < N$ for the spectral projections into the gaps of the Hofstadter operator [6].

Now, the subbundle $L_{q,r}(P) \subset E_{N,q}$ determined by the projection valued section $\Pi_{q,r}(P) = P(\cdot)$, for a projection $P \in \mathcal{A}_{q,r}^\theta$, will have as fiber over $k \in \mathbb{T}^2$ the space

$$L_{q,r}(P)|_k = \text{Range}(P(k)) \subset \mathcal{H}_{q,r}(k). \quad (17)$$

We need a dual bundle representation, $\Pi_{q,r}^{\text{ref}}$ of $\mathcal{A}_{q,r}^\theta$, s.t.

$$P(k_1, Nk_2) = P^{\text{ref}}(k_1, M_0 k_2), \quad (18)$$

and $P^{\text{ref}}(\cdot) = \Pi_{q,r}^{\text{ref}}(P)$. We are lead to the representation

$$U_q^{\text{ref}}(k) = e^{i2\pi k_2} (\mathbb{U}_N)^{qM}, \quad V_{q,r}^{\text{ref}}(k) = V_{q,r}^\theta(k) = e^{i2\pi n_r k_1} (\mathbb{V}_{N;k_1})^{d_r}. \quad (19)$$

which obey $U_q^{\text{ref}}(\cdot) V_{q,r}^{\text{ref}}(\cdot) = e^{i2\pi \frac{M}{N}} V_{q,r}^{\text{ref}}(\cdot) U_q^{\text{ref}}(\cdot)$. As elements in $C(\mathbb{T}^2) \otimes \text{Mat}_N(\mathbb{C}) \simeq C(\mathbb{T}^2; \text{Mat}_N(\mathbb{C}))$ they yield a representation of $\mathcal{A}_{q,r}^\theta$ as endomorphisms of the the trivial bundle $\mathbb{T}^2 \times \mathbb{C}^N \rightarrow \mathbb{T}^2$. Then, any projection P in $\mathcal{A}_{q,r}^\theta$ is mapped to a projection-valued section $P^{\text{ref}}(\cdot) = \Pi_{q,r}^{\text{ref}}(p)$ which defines a vector subbundle $L_{\text{ref}}(p) \rightarrow \mathbb{T}^2$ of the trivial vector bundle $\mathbb{T}^2 \times \mathbb{C}^N$. It will have as fiber over $k \in \mathbb{T}^2$ the space

$$L_{\text{ref}}(P)|_k = \text{Range}(P^{\text{ref}}(k)) \subset \mathbb{C}^N. \quad (20)$$

Then, equation (18) say that the vector bundle $L_{q,r}(P)$ “winded” around N times in the second direction is (locally) isomorphic to the vector bundle $L_{\text{ref}}(P)$ “winded” around M_0 times in the same direction. There is however an extra twist, due to the bundle $E_{N,q}$, of which $L_{q,r}(P)$ is a subbundle, being not trivial. Indeed, an analysis of the transition functions lead to the bundle isomorphism

$$\varphi_{(1,N)}^* L_{q,r}(P) \simeq \varphi_{(1,M_0)}^* L_{\text{ref}}(P) \otimes \det(E_{N,q}). \quad (21)$$

Here $\det(E_{N,q}) \rightarrow \mathbb{T}^2$ is the determinant line bundle and the extra operation $\varphi_{(1,N)}^*$ (the pullback) stays for the extra winding by N (for the bundle $L_{q,r}(P)$) and the same for $\varphi_{(1,M_0)}^*$ (for the bundle $L_{\text{ref}}(P)$). Formula (13) is the relation among corresponding first Chern numbers. Using the fact that $C_1(\varphi_{(1,N)}^* L_{q,r}(P)) = NC_1(L_{q,r}(P))$ and $C_1(\varphi_{(1,M_0)}^* L_{\text{ref}}(P)) = M_0 C_1(L_{\text{ref}}(P))$, as well as the identity $C_1(\det(E_{N,q})) = C_1(E_{N,q}) = q$, the relation (13) follows from (21) by standard arguments.

THE IRRATIONAL CASE

On the algebra $\mathcal{A}_{q,r}^\theta$ there is a faithful trace defined by

$$\tau \left((U_q)^n (V_{q,r}^\theta)^m \right) = \delta_{n,0} \delta_{m,0}$$

on monomials, and extended by linearity. Derivations $\partial_j : \mathcal{A}_{q,r}^\theta \rightarrow \mathcal{A}_{q,r}^\theta$, for $j = 1, 2$, defined on monomials by

$$\partial_1((U_q)^n (V_{q,r}^\theta)^m) = i 2\pi n (U_q)^n (V_{q,r}^\theta)^m, \quad \partial_2((U_q)^n (V_{q,r}^\theta)^m) = i 2\pi m (U_q)^n (V_{q,r}^\theta)^m,$$

are extended by linearity and Leibniz rule. Lastly, we need the first Connes-Chern number which, for a projection $P \in \mathcal{A}_{q,r}^\theta$ (in the domain of the derivations) computes the integer (an index of a Fredholm operator)

$$\mathcal{C}_1(P) = \frac{1}{i 2\pi} \tau(P(\partial_1(P)\partial_2(P) - \partial_2(P)\partial_1(P))).$$

Let $H_{q,r}^\theta \in \mathcal{A}_{q,r}^\theta$ be the Hofstadter operator (5) with associated spectral projection P_g^θ for the gap g as in (12). For $\theta \in I \subset \mathbb{R}$, the functional expression of $H_{q,r}^\theta \in \mathcal{A}_{q,r}^\theta$ is fixed and $H_{q,r}^\theta$ depends on the parameter θ only through the fundamental commutation relation which defines $\mathcal{A}_{q,r}^\theta$. Now, if the gap g is open for all $\theta \in I$ (with I sufficiently small), the functions $\theta \mapsto \mathcal{C}_1(P_g^\theta)$ is constant in the interval I [5]. On the other hand, from the structure of the group $K_0(\mathcal{A}_{q,r}^\theta)$, one deduces [18, 7] that

$$\tau(P_g^\theta) = m P_g^\theta - \theta \mathcal{C}_1(P_g^\theta), \tag{22}$$

with the integer $m(\cdot) \in \mathbb{Z}$ uniquely determined by the condition $0 \leq \tau(\cdot) \leq 1$. From (22), the integer $m(\cdot)$ is constant for $\theta \in I$. Hence, formula

$$C_{q,r}(P_g) = q \left[m(P_g) - \frac{r}{q} \mathcal{C}_1(P_g) \right] = q \left[\tau(P_g) + \left(\theta - \frac{r}{q} \right) \mathcal{C}_1(P_g) \right] \in \mathbb{Z} \tag{23}$$

is well defined and extends (13) for irrational values $\theta \in I$ (for which the gap g remains open). Indeed, for a rational $\theta = M/N$ one has natural identifications $\text{Rk}(L_{\text{ref}}(P)) = \tau(P)$ and $C_1(L_{\text{ref}}(P)) = \mathcal{C}_1(P)$ [10] and for the rational torus, formula (23) is the same as (13).

We think of (23) as relating conductances for the Harper operator $H_{q,r}^\theta$ in (5), thus generalizing (14) to

$$t_g + (q\theta - r)s_g = qd_g, \tag{24}$$

with P_g once again the spectral projections of the Harper operator $H_{q,r}^\theta$, and now identifying $t_g = C_{q,r}(P_g)$ and $s_g = -\mathcal{C}_1(P_g)$ as before, whereas $d_g = \tau(P_g)$.

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